

Comparison Theorem, Feynman-Kac Formula and Girsanov Transformation for BSDEs Driven by G -Brownian Motion

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Abstract

In this paper, we study comparison theorem, nonlinear Feynman-Kac formula and Girsanov transformation of the following BSDE driven by a G -Brownian motion.

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds + \int_t^T g(s, Y_s, Z_s) d\langle B \rangle_s \\ - \int_t^T Z_s dB_s - (K_T - K_t),$$

where K is a decreasing G -martingale.

Key words: G -expectation, Backward SDEs, Comparison theorem, Feynman-Kac formula, Girsanov transformation

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1 Introduction

Recently, Peng systemically established a time-consistent fully nonlinear expectation theory (see [17], [18] and [23]).

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As a typical and important case, Peng (2006) introduced the G -expectation theory (see [24] and the references therein). In the G -expectation framework (G -framework for short), the notion of G -Brownian motion and the corresponding stochastic calculus of Itô's type were established.

The solution of a BSDE driven by G -Brownian motion consists of a triple of processes (Y, Z, K) , satisfying

$$\begin{aligned} Y_t = & \xi + \int_t^T f(s, Y_s, Z_s) ds + \int_t^T g(s, Y_s, Z_s) d\langle B \rangle_s \\ & - \int_t^T Z_s dB_s - (K_T - K_t). \end{aligned} \quad (1.1)$$

The existence and uniqueness of the solution (Y, Z, K) for (1.1) is proved in [7]. In this paper, we further consider the related topics associated with this kind of G -BSDEs.

We first study the comparison theorem which is one of the most important properties of BSDEs. In order to prove this theorem, the explicit solutions of linear G -BSDEs are obtained. In order to do this it seems that we have to define the dual forward equations in an extended G -expectation space if the linear G -BSDEs include the ds term. The Gronwall inequality is derived as a by-product which is interesting by itself.

Then we explore the link between G -BSDEs and partial differential equations (PDE for short). It is well known that under a strong elliptic assumption, Peng [13] established a probabilistic interpretation of a system of quasi-linear PDEs via classical BSDEs. Then Peng [15] and Pardoux & Peng [14] obtained this interpretation for possibly degenerate situation. This interpretation establishes a one to one correspondence between the solution of a PDE and the corresponding classical BSDE, i.e. the so-called nonlinear Feynman-Kac formula. Peng gave the nonlinear Feynman-Kac Formula for a special type of G -BSDEs in [24]. In this paper, we consider the following type of G -FBSDEs:

$$dX_s^{t,\xi} = b(s, X_s^{t,\xi}) ds + h_{ij}(s, X_s^{t,\xi}) d\langle B^i, B^j \rangle_s + \sigma_j(s, X_s^{t,\xi}) dB_s^j, \quad X_t^{t,\xi} = \xi,$$

$$\begin{aligned} Y_s^{t,\xi} = & \Phi(X_T^{t,\xi}) + \int_s^T f(r, X_r^{t,\xi}, Y_r^{t,\xi}, Z_r^{t,\xi}) dr + \int_s^T g_{ij}(r, X_r^{t,\xi}, Y_r^{t,\xi}, Z_r^{t,\xi}) d\langle B^i, B^j \rangle_r \\ & - \int_s^T Z_r^{t,\xi} dB_r - (K_T^{t,\xi} - K_s^{t,\xi}), \end{aligned}$$

Set $u(t, x) := Y_t^{t,x}$. We prove that $u(t, x)$ is the unique viscosity solution of the following PDE:

$$\begin{cases} \partial_t u + F(D_x^2 u, D_x u, u, x, t) = 0, \\ u(T, x) = \Phi(x), \end{cases}$$

where

$$\begin{aligned} F(D_x^2 u, D_x u, u, x, t) = & G(H(D_x^2 u, D_x u, u, x, t)) + \langle b(t, x), D_x u \rangle \\ & + f(t, x, u, \langle \sigma_1(t, x), D_x u \rangle, \dots, \langle \sigma_d(t, x), D_x u \rangle), \end{aligned}$$

$$H_{ij}(D_x^2 u, D_x u, u, x, t) = \langle D_x^2 u \sigma_i(t, x), \sigma_j(t, x) \rangle + 2 \langle D_x u, h_{ij}(t, x) \rangle \\ + 2g_{ij}(t, x, u, \langle \sigma_1(t, x), D_x u \rangle, \dots, \langle \sigma_d(t, x), D_x u \rangle).$$

Finally, we study the Girsanov transformation. Different from [11] and [31], we discuss the Girsanov transformation of the following form:

$$\bar{B}_t := B_t - \int_0^t b_s ds - \int_0^t d_s^{ij} d\langle B^i, B^j \rangle_s.$$

We give a direct and simple method to prove that \bar{B}_t is a G -Brownian motion under a consistent sublinear expectation.

The paper is organized as follows. In section 2, we present some preliminaries for stochastic calculus under G -framework. The explicit solutions of linear G -BSDEs and the comparison theorem are established in section 3. In section 4, we obtain the nonlinear Feynman-Kac formula for a fully nonlinear PDE. We prove the Girsanov transformation for G -Brownian motion in section 5.

2 Preliminaries

We review some basic notions and results of G -expectation, the related spaces of random variables and the backward stochastic differential equations driven by a G -Brownian motion. The readers may refer to [7], [19], [20], [21], [22], [24] for more details.

Definition 2.1 *Let Ω be a given set and let \mathcal{H} be a vector lattice of real valued functions defined on Ω , namely $c \in \mathcal{H}$ for each constant c and $|X| \in \mathcal{H}$ if $X \in \mathcal{H}$. \mathcal{H} is considered as the space of random variables. A sublinear expectation $\hat{\mathbb{E}}$ on \mathcal{H} is a functional $\hat{\mathbb{E}} : \mathcal{H} \rightarrow \mathbb{R}$ satisfying the following properties: for all $X, Y \in \mathcal{H}$, we have*

- (a) *Monotonicity: If $X \geq Y$ then $\hat{\mathbb{E}}[X] \geq \hat{\mathbb{E}}[Y]$;*
- (b) *Constant preservation: $\hat{\mathbb{E}}[c] = c$;*
- (c) *Sub-additivity: $\hat{\mathbb{E}}[X + Y] \leq \hat{\mathbb{E}}[X] + \hat{\mathbb{E}}[Y]$;*
- (d) *Positive homogeneity: $\hat{\mathbb{E}}[\lambda X] = \lambda \hat{\mathbb{E}}[X]$ for each $\lambda \geq 0$.*

$(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ is called a sublinear expectation space.

Definition 2.2 *Let X_1 and X_2 be two n -dimensional random vectors defined respectively in sublinear expectation spaces $(\Omega_1, \mathcal{H}_1, \hat{\mathbb{E}}_1)$ and $(\Omega_2, \mathcal{H}_2, \hat{\mathbb{E}}_2)$. They are called identically distributed, denoted by $X_1 \stackrel{d}{=} X_2$, if $\hat{\mathbb{E}}_1[\varphi(X_1)] = \hat{\mathbb{E}}_2[\varphi(X_2)]$, for all $\varphi \in C_{l,Lip}(\mathbb{R}^n)$, where $C_{l,Lip}(\mathbb{R}^n)$ is the space of real continuous functions defined on \mathbb{R}^n such that*

$$|\varphi(x) - \varphi(y)| \leq C(1 + |x|^k + |y|^k)|x - y| \quad \text{for all } x, y \in \mathbb{R}^n,$$

where k and C depend only on φ .

Definition 2.3 In a sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$, a random vector $Y = (Y_1, \dots, Y_n)$, $Y_i \in \mathcal{H}$, is said to be independent of another random vector $X = (X_1, \dots, X_m)$, $X_i \in \mathcal{H}$ under $\hat{\mathbb{E}}[\cdot]$, denoted by $Y \perp X$, if for every test function $\varphi \in C_{l.Lip}(\mathbb{R}^m \times \mathbb{R}^n)$ we have $\hat{\mathbb{E}}[\varphi(X, Y)] = \hat{\mathbb{E}}[\hat{\mathbb{E}}[\varphi(x, Y)]_{x=X}]$.

Definition 2.4 (*G-normal distribution*) A d -dimensional random vector $X = (X_1, \dots, X_d)$ in a sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ is called *G-normally distributed* if for each $a, b \geq 0$ we have

$$aX + b\bar{X} \stackrel{d}{=} \sqrt{a^2 + b^2}X,$$

where \bar{X} is an independent copy of X , i.e., $\bar{X} \stackrel{d}{=} X$ and $\bar{X} \perp X$. Here the letter G denotes the function

$$G(A) := \frac{1}{2} \hat{\mathbb{E}}[\langle AX, X \rangle] : \mathbb{S}_d \rightarrow \mathbb{R},$$

where \mathbb{S}_d denotes the collection of $d \times d$ symmetric matrices.

Peng [22] showed that $X = (X_1, \dots, X_d)$ is *G-normally distributed* if and only if for each $\varphi \in C_{l.Lip}(\mathbb{R}^d)$, $u(t, x) := \hat{\mathbb{E}}[\varphi(x + \sqrt{t}X)]$, $(t, x) \in [0, \infty) \times \mathbb{R}^d$, is the solution of the following *G-heat equation*:

$$\partial_t u - G(D_x^2 u) = 0, \quad u(0, x) = \varphi(x).$$

The function $G(\cdot) : \mathbb{S}_d \rightarrow \mathbb{R}$ is a monotonic, sublinear mapping on \mathbb{S}_d and $G(A) = \frac{1}{2} \hat{\mathbb{E}}[\langle AX, X \rangle] \leq \frac{1}{2} |A| \hat{\mathbb{E}}[|X|^2] =: \frac{1}{2} |A| \bar{\sigma}^2$ implies that there exists a bounded, convex and closed subset $\Gamma \subset \mathbb{S}_d^+$ such that

$$G(A) = \frac{1}{2} \sup_{\gamma \in \Gamma} \text{tr}[\gamma A],$$

where \mathbb{S}_d^+ denotes the collection of nonnegative elements in \mathbb{S}_d .

In this paper, we only consider non-degenerate *G-normal distribution*, i.e., there exists some $\underline{\sigma}^2 > 0$ such that $G(A) - G(B) \geq \underline{\sigma}^2 \text{tr}[A - B]$ for any $A \geq B$.

Definition 2.5 i) Let $\Omega_T = C_0([0, T]; \mathbb{R}^d)$, the space of real valued continuous functions on $[0, T]$ with $\omega_0 = 0$, be endowed with the supremum norm and let $B_t(\omega) = \omega_t$ be the canonical process. Set

$$\mathcal{H}_T^0 := \{\varphi(B_{t_1}, \dots, B_{t_n}) : n \geq 1, t_1, \dots, t_n \in [0, T], \varphi \in C_{l.Lip}(\mathbb{R}^{d \times n})\}.$$

Let $G : \mathbb{S}_d \rightarrow \mathbb{R}$ be a given monotonic and sublinear function. *G-expectation* is a sublinear expectation defined by

$$\hat{\mathbb{E}}[X] = \tilde{\mathbb{E}}[\varphi(\sqrt{t_1 - t_0} \xi_1, \dots, \sqrt{t_m - t_{m-1}} \xi_m)],$$

for all $X = \varphi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \dots, B_{t_m} - B_{t_{m-1}})$, where ξ_1, \dots, ξ_n are identically distributed d -dimensional *G-normally distributed* random vectors in a sublinear expectation space $(\tilde{\Omega}, \tilde{\mathcal{H}}, \tilde{\mathbb{E}})$ such that ξ_{i+1} is independent of (ξ_1, \dots, ξ_i)

for every $i = 1, \dots, m-1$. The corresponding canonical process $B_t = (B_t^i)_{i=1}^d$ is called a G -Brownian motion.

ii) Let us define the conditional G -expectation $\hat{\mathbb{E}}_t$ of $\xi \in \mathcal{H}_T^0$ knowing \mathcal{H}_t^0 , for $t \in [0, T]$. Without loss of generality we can assume that ξ has the representation $\xi = \varphi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \dots, B_{t_m} - B_{t_{m-1}})$ with $t = t_i$, for some $1 \leq i \leq m$, and we put

$$\begin{aligned} \hat{\mathbb{E}}_{t_i}[\varphi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \dots, B_{t_m} - B_{t_{m-1}})] \\ = \tilde{\varphi}(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \dots, B_{t_i} - B_{t_{i-1}}), \end{aligned}$$

where

$$\tilde{\varphi}(x_1, \dots, x_i) = \hat{\mathbb{E}}[\varphi(x_1, \dots, x_i, B_{t_{i+1}} - B_{t_i}, \dots, B_{t_m} - B_{t_{m-1}})].$$

Define $\|\xi\|_{p,G} = (\hat{\mathbb{E}}[|\xi|^p])^{1/p}$ for $\xi \in \mathcal{H}_T^0$ and $p \geq 1$. Then for all $t \in [0, T]$, $\hat{\mathbb{E}}_t[\cdot]$ is a continuous mapping on \mathcal{H}_T^0 w.r.t. the norm $\|\cdot\|_{1,G}$. Therefore it can be extended continuously to the completion $L_G^1(\Omega_T)$ of \mathcal{H}_T^0 under the norm $\|\cdot\|_{1,G}$.

Let $L_{ip}(\Omega_T) := \{\varphi(B_{t_1}, \dots, B_{t_n}) : n \geq 1, t_1, \dots, t_n \in [0, T], \varphi \in C_{b.Lip}(\mathbb{R}^{d \times n})\}$, where $C_{b.Lip}(\mathbb{R}^{d \times n})$ denotes the set of bounded Lipschitz functions on $\mathbb{R}^{d \times n}$. Denis et al. [5] proved that the completions of $C_b(\Omega_T)$ (the set of bounded continuous function on Ω_T), \mathcal{H}_T^0 and $L_{ip}(\Omega_T)$ under $\|\cdot\|_{p,G}$ are the same and we denote them by $L_G^p(\Omega_T)$.

For each fixed $\mathbf{a} \in \mathbb{R}^d$, $B_t^{\mathbf{a}} = \langle \mathbf{a}, B_t \rangle$ is a 1-dimensional $G_{\mathbf{a}}$ -Brownian motion, where $G_{\mathbf{a}}(\alpha) = \frac{1}{2}(\sigma_{\mathbf{a}\mathbf{a}^T}^2 \alpha^+ - \sigma_{-\mathbf{a}\mathbf{a}^T}^2 \alpha^-)$, $\sigma_{\mathbf{a}\mathbf{a}^T}^2 = 2G(\mathbf{a}\mathbf{a}^T)$, $\sigma_{-\mathbf{a}\mathbf{a}^T}^2 = -2G(-\mathbf{a}\mathbf{a}^T)$. Let $\pi_t^N = \{t_0^N, \dots, t_N^N\}$, $N = 1, 2, \dots$, be a sequence of partitions of $[0, t]$ such that $\mu(\pi_t^N) = \max\{|t_{i+1}^N - t_i^N| : i = 0, \dots, N-1\} \rightarrow 0$, the quadratic variation process of $B^{\mathbf{a}}$ is defined by

$$\langle B^{\mathbf{a}} \rangle_t = \lim_{\mu(\pi_t^N) \rightarrow 0} \sum_{j=0}^{N-1} (B_{t_{j+1}^N}^{\mathbf{a}} - B_{t_j^N}^{\mathbf{a}})^2.$$

For each fixed $\mathbf{a}, \bar{\mathbf{a}} \in \mathbb{R}^d$, the mutual variation process of $B^{\mathbf{a}}$ and $B^{\bar{\mathbf{a}}}$ is defined by

$$\langle B^{\mathbf{a}}, B^{\bar{\mathbf{a}}} \rangle_t = \frac{1}{4}[\langle B^{\mathbf{a}+\bar{\mathbf{a}}} \rangle_t - \langle B^{\mathbf{a}-\bar{\mathbf{a}}} \rangle_t].$$

Definition 2.6 Let $M_G^0(0, T)$ be the collection of processes in the following form: for a given partition $\{t_0, \dots, t_N\} = \pi_T$ of $[0, T]$,

$$\eta_t(\omega) = \sum_{j=0}^{N-1} \xi_j(\omega) I_{[t_j, t_{j+1})}(t),$$

where $\xi_i \in L_{ip}(\Omega_{t_i})$, $i = 0, 1, 2, \dots, N-1$. For $p \geq 1$ and $\eta \in M_G^0(0, T)$, let $\|\eta\|_{H_G^p} = \{\hat{\mathbb{E}}[(\int_0^T |\eta_s|^2 ds)^{p/2}]\}^{1/p}$, $\|\eta\|_{M_G^p} = \{\hat{\mathbb{E}}[\int_0^T |\eta_s|^p ds]\}^{1/p}$ and denote by $H_G^p(0, T)$, $M_G^p(0, T)$ the completions of $M_G^0(0, T)$ under the norms $\|\cdot\|_{H_G^p}$, $\|\cdot\|_{M_G^p}$ respectively.

Theorem 2.7 ([5, 8]) *There exists a weakly compact set $\mathcal{P} \subset \mathcal{M}_1(\Omega_T)$, the set of probability measures on $(\Omega_T, \mathcal{B}(\Omega_T))$, such that*

$$\hat{\mathbb{E}}[\xi] = \sup_{P \in \mathcal{P}} E_P[\xi] \quad \text{for all } \xi \in \mathcal{H}_T^0.$$

\mathcal{P} is called a set that represents $\hat{\mathbb{E}}$.

Let \mathcal{P} be a weakly compact set that represents $\hat{\mathbb{E}}$. For this \mathcal{P} , we define capacity

$$c(A) := \sup_{P \in \mathcal{P}} P(A), \quad A \in \mathcal{B}(\Omega_T).$$

A set $A \subset \Omega_T$ is polar if $c(A) = 0$. A property holds “quasi-surely” (q.s. for short) if it holds outside a polar set. In the following, we do not distinguish two random variables X and Y if $X = Y$ q.s.. We set

$$\mathbb{L}^p(\Omega_t) := \{X \in \mathcal{B}(\Omega_t) : \sup_{P \in \mathcal{P}} E_P[|X|^p] < \infty\} \text{ for } p \geq 1.$$

It is important to note that $L_G^p(\Omega_t) \subset \mathbb{L}^p(\Omega_t)$. We extend G -expectation $\hat{\mathbb{E}}$ to $\mathbb{L}^p(\Omega_t)$ and still denote it by $\hat{\mathbb{E}}$, for each $X \in \mathbb{L}^1(\Omega_T)$, we set

$$\hat{\mathbb{E}}[X] = \sup_{P \in \mathcal{P}} E_P[X].$$

For $p \geq 1$, $\mathbb{L}^p(\Omega_t)$ is a Banach space under the norm $(\hat{\mathbb{E}}[|\cdot|^p])^{1/p}$.

Set

$$\mathbb{L}_G^{0,p,t}(\Omega_T) := \{\xi = \sum_{i=1}^n \eta_i I_{A_i} : A_i \in \mathcal{B}(\Omega_t), \eta_i \in L_G^p(\Omega), n \in \mathbb{N}\},$$

we define the corresponding conditional G -expectation, still denoted by $\hat{\mathbb{E}}_s[\cdot]$, by setting

$$\hat{\mathbb{E}}_s[\sum_{i=1}^n \eta_i I_{A_i}] := \sum_{i=1}^n \hat{\mathbb{E}}_s[\eta_i] I_{A_i} \quad \text{for } s \geq t.$$

Proposition 2.8 ([7]) *For each $\xi, \eta \in \mathbb{L}_G^{0,1,t}(\Omega_T)$, we have*

- (i) *Monotonicity: If $\xi \leq \eta$, then $\hat{\mathbb{E}}_s[\xi] \leq \hat{\mathbb{E}}_s[\eta]$ for any $s \geq t$;*
- (ii) *Constant preserving: If $\xi \in \mathbb{L}_G^{0,1,t}(\Omega_t)$, then $\hat{\mathbb{E}}_t[\xi] = \xi$;*
- (iii) *Sub-additivity: $\hat{\mathbb{E}}_s[\xi + \eta] \leq \hat{\mathbb{E}}_s[\xi] + \hat{\mathbb{E}}_s[\eta]$ for any $s \geq t$;*
- (iv) *Positive homogeneity: If $\xi \in \mathbb{L}_G^{0,\infty,t}(\Omega_t)$ and $\xi \geq 0$, then $\hat{\mathbb{E}}_t[\xi\eta] = \xi\hat{\mathbb{E}}_t[\eta]$;*
- (v) *Consistency: For $t \leq s \leq r$, we have $\hat{\mathbb{E}}_s[\hat{\mathbb{E}}_r[\xi]] = \hat{\mathbb{E}}_s[\xi]$.*
- (vi) $\hat{\mathbb{E}}[\hat{\mathbb{E}}_t[\xi]] = \hat{\mathbb{E}}[\xi].$

Let $\mathbb{L}_G^{p,t}(\Omega_T)$ be the completion of $\mathbb{L}_G^{0,p,t}(\Omega_T)$ under the norm $(\hat{\mathbb{E}}[\|\cdot\|^p])^{1/p}$. Clearly, the conditional G -expectation can be extended continuously to $\mathbb{L}_G^{1,t}(\Omega_T)$.

Set

$$\mathbb{M}^{p,0}(0, T) := \{\eta_t = \sum_{i=0}^{N-1} \xi_{t_i} I_{[t_i, t_{i+1})}(t) : 0 = t_0 < \dots < t_N = T, \xi_{t_i} \in \mathbb{L}^p(\Omega_{t_i})\}.$$

For $p \geq 1$, we denote by $\mathbb{M}^p(0, T)$, $\mathbb{H}^p(0, T)$, $\mathbb{S}^p(0, T)$ the completion of $\mathbb{M}^{p,0}(0, T)$ under the norm $\|\eta\|_{\mathbb{M}^p} := (\hat{\mathbb{E}}[\int_0^T |\eta_t|^p dt])^{1/p}$, $\|\eta\|_{\mathbb{H}^p} := \{\hat{\mathbb{E}}[(\int_0^T |\eta_t|^2 dt)^{p/2}]\}^{1/p}$, $\|\eta\|_{\mathbb{S}^p} := (\hat{\mathbb{E}}[\sup_{t \in [0, T]} |\eta_t|^p])^{1/p}$ respectively. Following Li and Peng [10], for each $\eta \in \mathbb{H}^p(0, T)$ with $p \geq 1$, we can define Itô's integral $\int_0^T \eta_s dB_s$. Moreover, by Proposition 2.10 in [10] and classical Burkholder-Davis-Gundy Inequality, the following properties hold.

Proposition 2.9 *For each $\eta, \theta \in \mathbb{H}^\alpha(0, T)$ with $\alpha \geq 1$ and $p > 0$, $\xi \in \mathbb{L}^\infty(\Omega_t)$, we have*

$$\begin{aligned} \hat{\mathbb{E}}[\int_0^T \eta_s dB_s] &= 0, \\ \underline{c}^p c_p \hat{\mathbb{E}}[(\int_0^T |\eta_s|^2 ds)^{p/2}] &\leq \hat{\mathbb{E}}[\sup_{t \in [0, T]} |\int_0^t \eta_s dB_s|^p] \leq \bar{\sigma}^p C_p \hat{\mathbb{E}}[(\int_0^T |\eta_s|^2 ds)^{p/2}], \\ \int_t^T (\xi \eta_s + \theta_s) dB_s &= \xi \int_t^T \eta_s dB_s + \int_t^T \theta_s dB_s, \end{aligned}$$

where $0 < c_p < C_p < \infty$ are constants.

Remark 2.10 *If $\eta \in H_G^\alpha(0, T)$ with $\alpha \geq 1$ and $p \in (0, \alpha]$, then we can get $\sup_{u \in [t, T]} |\int_t^u \eta_s dB_s|^p \in L_G^1(\Omega_T)$ and*

$$\underline{c}^p c_p \hat{\mathbb{E}}_t[(\int_t^T |\eta_s|^2 ds)^{p/2}] \leq \hat{\mathbb{E}}_t[\sup_{u \in [t, T]} |\int_t^u \eta_s dB_s|^p] \leq \bar{\sigma}^p C_p \hat{\mathbb{E}}_t[(\int_t^T |\eta_s|^2 ds)^{p/2}].$$

Definition 2.11 *A process $\{M_t\}$ with values in $L_G^1(\Omega_T)$ is called a G -martingale if $\hat{\mathbb{E}}_s[M_t] = M_s$ for any $s \leq t$.*

Let $S_G^0(0, T) = \{h(t, B_{t_1 \wedge t}, \dots, B_{t_n \wedge t}) : t_1, \dots, t_n \in [0, T], h \in C_{b, Lip}(\mathbb{R}^{n+1})\}$. For $p \geq 1$ and $\eta \in S_G^0(0, T)$, set $\|\eta\|_{S_G^p} = \{\hat{\mathbb{E}}[\sup_{t \in [0, T]} |\eta_t|^p]\}^{\frac{1}{p}}$. Denote by $S_G^p(0, T)$ the completion of $S_G^0(0, T)$ under the norm $\|\cdot\|_{S_G^p}$.

We consider the following type of G -BSDEs (in this paper we always use Einstein convention):

$$\begin{aligned} Y_t &= \xi + \int_t^T f(s, Y_s, Z_s) ds + \int_t^T g_{ij}(s, Y_s, Z_s) d\langle B^i, B^j \rangle_s \\ &\quad - \int_t^T Z_s dB_s - (K_T - K_t), \end{aligned} \tag{2.1}$$

where

$$f(t, \omega, y, z), g_{ij}(t, \omega, y, z) : [0, T] \times \Omega_T \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$$

satisfy the following properties:

(H1) There exists some $\beta > 1$ such that for any $y, z, f(\cdot, \cdot, y, z), g_{ij}(\cdot, \cdot, y, z) \in M_G^\beta(0, T)$;

(H2) There exists some $L > 0$ such that

$$|f(t, y, z) - f(t, y', z')| + \sum_{i,j=1}^d |g_{ij}(t, y, z) - g_{ij}(t, y', z')| \leq L(|y - y'| + |z - z'|).$$

For simplicity, we denote by $\mathfrak{S}_G^\alpha(0, T)$ the collection of processes (Y, Z, K) such that $Y \in S_G^\alpha(0, T)$, $Z \in H_G^\alpha(0, T; \mathbb{R}^d)$, K is a decreasing G -martingale with $K_0 = 0$ and $K_T \in L_G^\alpha(\Omega_T)$.

Definition 2.12 Let $\xi \in L_G^\beta(\Omega_T)$ and f satisfy (H1) and (H2) for some $\beta > 1$. A triplet of processes (Y, Z, K) is called a solution of equation (2.1) if for some $1 < \alpha \leq \beta$ the following properties hold:

- (a) $(Y, Z, K) \in \mathfrak{S}_G^\alpha(0, T)$;
- (b) $Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds + \int_t^T g_{ij}(s, Y_s, Z_s) d\langle B^i, B^j \rangle_s - \int_t^T Z_s dB_s - (K_T - K_t)$.

Theorem 2.13 ([7]) Assume that $\xi \in L_G^\beta(\Omega_T)$ and f, g_{ij} satisfy (H1) and (H2) for some $\beta > 1$. Then equation (2.1) has a unique solution (Y, Z, K) . Moreover, for any $1 < \alpha < \beta$ we have $Y \in S_G^\alpha(0, T)$, $Z \in H_G^\alpha(0, T; \mathbb{R}^d)$ and $K_T \in L_G^\alpha(\Omega_T)$.

We have the following estimates.

Proposition 2.14 ([7]) Let $\xi \in L_G^\beta(\Omega_T)$ and f, g_{ij} satisfy (H1) and (H2) for some $\beta > 1$. Assume that $(Y, Z, K) \in \mathfrak{S}_G^\alpha(0, T)$ for some $1 < \alpha < \beta$ is a solution of equation (2.1). Then

- (i) There exists a constant $C_\alpha := C(\alpha, T, G, L) > 0$ such that

$$|Y_t|^\alpha \leq C_\alpha \hat{\mathbb{E}}_t[|\xi|^\alpha + \int_t^T |h_s^0|^\alpha ds],$$

$$\hat{\mathbb{E}}[(\int_0^T |Z_s|^2 ds)^{\frac{\alpha}{2}}] \leq C_\alpha \{ \hat{\mathbb{E}}[\sup_{t \in [0, T]} |Y_t|^\alpha] + (\hat{\mathbb{E}}[\sup_{t \in [0, T]} |Y_t|^\alpha])^{\frac{1}{2}} (\hat{\mathbb{E}}[(\int_0^T |h_s^0|^\alpha ds)]^{\frac{1}{2}} \},$$

$$\hat{\mathbb{E}}[|K_T|^\alpha] \leq C_\alpha \{ \hat{\mathbb{E}}[\sup_{t \in [0, T]} |Y_t|^\alpha] + \hat{\mathbb{E}}[(\int_0^T |h_s^0|^\alpha ds)] \},$$

where $h_s^0 = |f(s, 0, 0)| + \sum_{i,j=1}^d |g_{ij}(s, 0, 0)|$.

- (ii) For any given α' with $\alpha < \alpha' < \beta$, there exists a constant $C_{\alpha, \alpha'}$ depending on α, α', T, G, L such that

$$\begin{aligned} \hat{\mathbb{E}}[\sup_{t \in [0, T]} |Y_t|^\alpha] &\leq C_{\alpha, \alpha'} \{ \hat{\mathbb{E}}[\sup_{t \in [0, T]} \hat{\mathbb{E}}_t[|\xi|^\alpha]] \\ &\quad + (\hat{\mathbb{E}}[\sup_{t \in [0, T]} \hat{\mathbb{E}}_t[(\int_0^T h_s^0 ds)^{\alpha'}]])^{\frac{\alpha}{\alpha'}} + \hat{\mathbb{E}}[\sup_{t \in [0, T]} \hat{\mathbb{E}}_t[(\int_0^T h_s^0 ds)^{\alpha'}]] \}. \end{aligned}$$

Proposition 2.15 ([7]) Let $\xi^l \in L_G^\beta(\Omega_T)$, $l = 1, 2$, and f^l, g_{ij}^l satisfy (H1) and (H2') for some $\beta > 1$. Assume that $(Y^l, Z^l, K^l) \in \mathfrak{S}_G^\alpha(0, T)$ for some $1 < \alpha < \beta$ are the solutions of equation (2.1) corresponding to ξ^l, f^l and g_{ij}^l . Set $\hat{Y}_t = Y_t^1 - Y_t^2, \hat{Z}_t = Z_t^1 - Z_t^2$ and $\hat{K}_t = K_t^1 - K_t^2$. Then

- (i) There exists a constant $C_\alpha := C(\alpha, T, G, L) > 0$ such that

$$|\hat{Y}_t|^\alpha \leq C_\alpha \hat{\mathbb{E}}_t[|\hat{\xi}|^\alpha + \int_t^T |\hat{h}_s|^\alpha ds],$$

where $\hat{\xi} = \xi^1 - \xi^2, \hat{h}_s = |f^1(s, Y_s^2, Z_s^2) - f^2(s, Y_s^2, Z_s^2)| + \sum_{i,j=1}^d |g_{ij}^1(s, Y_s^2, Z_s^2) - g_{ij}^2(s, Y_s^2, Z_s^2)|$.

- (ii) For any given α' with $\alpha < \alpha' < \beta$, there exists a constant $C_{\alpha, \alpha'}$ depending on α, α', T, G, L such that

$$\begin{aligned} \hat{\mathbb{E}}[\sup_{t \in [0, T]} |\hat{Y}_t|^\alpha] &\leq C_{\alpha, \alpha'} \{ \hat{\mathbb{E}}[\sup_{t \in [0, T]} \hat{\mathbb{E}}_t[|\hat{\xi}|^\alpha]] \\ &\quad + (\hat{\mathbb{E}}[\sup_{t \in [0, T]} \hat{\mathbb{E}}_t[(\int_0^T \hat{h}_s ds)^{\alpha'}]])^{\frac{\alpha}{\alpha'}} + \hat{\mathbb{E}}[\sup_{t \in [0, T]} \hat{\mathbb{E}}_t[(\int_0^T \hat{h}_s ds)^{\alpha'}]] \}. \end{aligned}$$

3 Comparison theorem of G -BSDEs

For simplicity, we consider 1-dimensional G -Brownian motion case. The results still hold for the case $d > 1$.

3.1 Explicit solutions of linear G -BSDEs

Let $(\Omega_T, L_G^1(\Omega_T), \hat{\mathbb{E}})$ with $\Omega_T = C_0([0, T], \mathbb{R})$ be a G -expectation space. We consider the explicit solution of the following linear G -BSDE:

$$Y_t = \xi + \int_t^T f_s ds + \int_t^T g_s d\langle B \rangle_s - \int_t^T Z_s dB_s - (K_T - K_t), \quad (3.1)$$

where $f_s = a_s Y_s + b_s Z_s + m_s, g_s = c_s Y_s + d_s Z_s + n_s$ with $\{a_s\}_{s \in [0, T]}, \{b_s\}_{s \in [0, T]}, \{c_s\}_{0 \leq s \in [0, T]}, \{d_s\}_{s \in [0, T]}$ bounded processes in $M_G^\beta(0, T)$ and $\xi \in L_G^\beta(\Omega_T), \{m_s\}_{s \in [0, T]}, \{n_s\}_{s \in [0, T]} \in M_G^\beta(0, T)$ with $\beta > 1$. For this purpose we construct an auxiliary extended \tilde{G} -expectation space $(\tilde{\Omega}_T, L_G^1(\tilde{\Omega}_T), \hat{\mathbb{E}}^{\tilde{G}})$ with $\tilde{\Omega}_T = C_0([0, T], \mathbb{R}^2)$ and

$$\tilde{G}(A) = \frac{1}{2} \sup_{\underline{\sigma}^2 \leq v \leq \bar{\sigma}^2} \text{tr} \left[A \begin{bmatrix} v & 1 \\ 1 & v^{-1} \end{bmatrix} \right], \quad A \in \mathbb{S}_2.$$

Let $\{(B_t, \tilde{B}_t)\}$ be the canonical process in the extended space.

Remark 3.1 *It is easy to check that $\langle B, \tilde{B} \rangle_t = t$. In particular, if $\underline{\sigma}^2 = \bar{\sigma}^2$, we can further get $\tilde{B}_t = \bar{\sigma}^{-2} B_t$.*

Let $\{X_t\}_{t \in [0, T]}$ be the solution of the following \tilde{G} -SDE:

$$X_t = 1 + \int_0^t a_s X_s ds + \int_0^t c_s X_s d\langle B \rangle_s + \int_0^t d_s X_s dB_s + \int_0^t b_s X_s d\tilde{B}_s. \quad (3.2)$$

It is easy to verify that

$$X_t = \exp\left(\int_0^t (a_s - b_s d_s) ds + \int_0^t c_s d\langle B \rangle_s\right) \mathcal{E}_t^B \mathcal{E}_t^{\tilde{B}}, \quad (3.3)$$

where $\mathcal{E}_t^B = \exp(\int_0^t d_s dB_s - \frac{1}{2} \int_0^t d_s^2 d\langle B \rangle_s)$, $\mathcal{E}_t^{\tilde{B}} = \exp(\int_0^t b_s d\tilde{B}_s - \frac{1}{2} \int_0^t b_s^2 d\langle \tilde{B} \rangle_s)$.

Theorem 3.2 *In the extended \tilde{G} -expectation space, the solution of the G -BSDE (3.1) can be represented as*

$$Y_t = (X_t)^{-1} \hat{\mathbb{E}}_t^{\tilde{G}}[X_T \xi + \int_t^T m_s X_s ds + \int_t^T n_s X_s d\langle B \rangle_s], \quad (3.4)$$

where $\{X_t\}_{t \in [0, T]}$ is the solution of the \tilde{G} -SDE (3.2).

Proof. By applying Itô's formula to $X_t Y_t$, we get

$$\begin{aligned} X_t Y_t &+ \int_t^T (X_s Z_s + d_s X_s Y_s) dB_s + \int_t^T b_s X_s Y_s d\tilde{B}_s + \int_t^T X_s dK_s \\ &= X_T \xi + \int_t^T m_s X_s ds + \int_t^T n_s X_s d\langle B \rangle_s. \end{aligned}$$

By Lemma 3.4 in [7], we have $\{\int_0^t X_s dK_s\}_{t \in [0, T]}$ is a \tilde{G} -martingale. Thus we get

$$Y_t = (X_t)^{-1} \hat{\mathbb{E}}_t^{\tilde{G}}[X_T \xi + \int_t^T m_s X_s ds + \int_t^T n_s X_s d\langle B \rangle_s].$$

□

Remark 3.3 *If $b_t = 0$, the solution of the G -BSDE (3.1) is*

$$Y_t = (X_t)^{-1} \hat{\mathbb{E}}_t[X_T \xi + \int_t^T m_s X_s ds + \int_t^T n_s X_s d\langle B \rangle_s],$$

where $X_t = \exp(\int_0^t a_s ds + \int_0^t (c_s - \frac{1}{2} d_s^2) d\langle B \rangle_s + \int_0^t d_s dB_s)$. In this case, we do not need to construct an auxiliary space. If $b_t \neq 0$, the form of X_t contains \tilde{B} , but

$$Y_t = \hat{\mathbb{E}}_t^{\tilde{G}}[X_T^t \xi + \int_t^T m_s X_s^t ds + \int_t^T n_s X_s^t d\langle B \rangle_s]$$

does not contain \tilde{B} , where $X_s^t = X_s/X_t$. For simplicity, we only give an explanation for $\xi = \varphi(B_T)$, $f_s = b_s Z_s$ with $b_s = \psi(B_s)$ and $g_s = 0$ in the G -BSDE (3.1). In this case,

$$\begin{aligned} Y_t &= \hat{\mathbb{E}}_t^{\tilde{G}}[\varphi(B_T) \exp(\int_t^T \psi(B_s) d\tilde{B}_s - \frac{1}{2} \int_t^T |\psi(B_s)|^2 d\langle \tilde{B} \rangle_s)] \\ &= \hat{\mathbb{E}}^{\tilde{G}}[\varphi(x + B_T^t) \exp(\int_t^T \psi(x + B_s^t) d\tilde{B}_s - \frac{1}{2} \int_t^T |\psi(x + B_s^t)|^2 d\langle \tilde{B} \rangle_s)]_{x=B_t}, \end{aligned}$$

which does not contain \tilde{B} , where $B_s^t = B_s - B_t$.

Note that $\hat{\mathbb{E}}_t^{\tilde{G}}[\xi] = \hat{\mathbb{E}}_t[\xi]$ for each $\xi \in L_G^1(\Omega_T)$, thus this Y in Theorem 3.2 is the solution of the G -BSDE (3.1) in $(\Omega_T, L_G^1(\Omega_T), \hat{\mathbb{E}})$. Here \tilde{B} is an auxiliary process and disappear by taking conditional expectation.

Remark 3.4 If $b_s = 0$, $d_s = 0$, we have the following special type of G -BSDE:

$$Y_t = \hat{\mathbb{E}}_t[\xi + \int_t^T (a_s Y_s + m_s) ds + \int_t^T (c_s Y_s + n_s) d\langle B \rangle_s], \quad (3.5)$$

where $\{a_s\}_{s \in [0, T]}$, $\{c_s\}_{s \in [0, T]}$ are bounded processes in $M_G^1(0, T)$ and $\xi \in L_G^1(\Omega)$, $\{m_s\}_{s \in [0, T]}$, $\{n_s\}_{s \in [0, T]} \in M_G^1(0, T)$. By applying Theorem 3.2 to $\xi^N = (\xi \wedge N) \vee (-N)$, $m_s^N = (m_s \wedge N) \vee (-N)$, $n_s^N = (n_s \wedge N) \vee (-N)$ for each $N > 0$, we obtain that the explicit solution of the G -BSDE (3.5) is

$$Y_t = (X_t)^{-1} \hat{\mathbb{E}}_t[X_T \xi + \int_t^T m_s X_s ds + \int_t^T n_s X_s d\langle B \rangle_s], \quad (3.6)$$

where $X_t = \exp(\int_0^t a_s ds + \int_0^t c_s d\langle B \rangle_s)$.

In the following, we explain why we have to extend the space. For simplicity, we only consider

$$Y_t = \xi + \int_t^T Z_s ds - \int_t^T Z_s dB_s - (K_T - K_t).$$

In order to get the explicit solution of the above G -BSDE, we try to find a positive process X (not depending on Y, Z, K) such that XY is a G -martingale. Applying Itô's formula to XY , we have

$$d(X_t Y_t) = X_t Z_t dB_t + X_t dK_t - X_t Z_t dt + Z_t d\langle X, B \rangle_t + Y_t dX_t.$$

So as to guarantee that XY is a G -martingale, $-X_t Z_t dt + Z_t d\langle X, B \rangle_t + Y_t dX_t$ should be a symmetric G -martingale, which implies that X is a symmetric G -martingale and

$$X_t dt = d\langle X, B \rangle_t. \quad (3.7)$$

By the representation theorem of symmetric G -martingales, we assume $X_t = X_0 + \int_0^t h_s dB_s$ for some $h \in M_G^2(0, T)$. Then equation (3.7) implies that

$$X_t dt = h_t d\langle B \rangle_t.$$

By Corollary 3.5 in [30], we have $X \equiv 0$ if $\underline{\sigma}^2 < \bar{\sigma}^2$. So generally we cannot find a proper process X in the original G -expectation space. Actually, in Theorem 3.2, we find a process X in the extended \tilde{G} -expectation space such that XY is a \tilde{G} -martingale instead of G -martingale.

Sometimes we say a process $Y \in S_G^\alpha(0, T)$ with some $\alpha > 1$ is a solution of equation (2.1) if there exist processes Z, K such that $(Y, Z, K) \in \mathfrak{S}_G^\alpha(0, T)$ is a solution of equation (2.1).

Proposition 3.5 *Let K be a decreasing G -martingale with $K_T \in L_G^\alpha(\Omega_T)$ for some $\alpha > 1$. Assume that*

$$f(t, K_t, 0) = g(t, K_t, 0) = 0.$$

Then K is a solution of equation (2.1).

Proof. It's easy to check that $(K, 0, K)$ is a solution of equation (2.1). \square

3.2 Comparison theorem of G -BSDEs

Theorem 3.6 *Let $(Y_t^i, Z_t^i, K_t^i)_{t \leq T}$, $i = 1, 2$, be the solutions of the following G -BSDEs:*

$$Y_t^i = \xi^i + \int_t^T f_i(s, Y_s^i, Z_s^i) ds + \int_t^T g_i(s, Y_s^i, Z_s^i) d\langle B \rangle_s - \int_t^T Z_s^i dB_s - (K_T^i - K_t^i),$$

where $\xi^i \in L_G^\beta(\Omega_T)$, f_i, g_i satisfy (H1) and (H2) with $\beta > 1$. If $\xi^1 \geq \xi^2$, $f_1 \geq f_2$, $g_1 \geq g_2$, then $Y_t^1 \geq Y_t^2$.

Proof. We have

$$\hat{Y}_t + K_t^2 = \hat{\xi} + K_T^2 + \int_t^T \hat{f}_s ds + \int_t^T \hat{g}_s d\langle B \rangle_s - \int_t^T \hat{Z}_s dB_s - (K_T^1 - K_t^1),$$

where $\hat{Y}_t = Y_t^1 - Y_t^2$, $\hat{Z}_t = Z_t^1 - Z_t^2$, $\hat{\xi} = \xi^1 - \xi^2 \geq 0$, $\hat{f}_s = f_1(s, Y_s^1, Z_s^1) - f_2(s, Y_s^2, Z_s^2)$, $\hat{g}_s = g_1(s, Y_s^1, Z_s^1) - g_2(s, Y_s^2, Z_s^2)$. For each given $\varepsilon > 0$, we can choose Lipschitz function $l(\cdot)$ such that $I_{[-\varepsilon, \varepsilon]} \leq l(x) \leq I_{[-2\varepsilon, 2\varepsilon]}$. Thus we have

$$f_1(s, Y_s^1, Z_s^1) - f_1(s, Y_s^2, Z_s^1) = (f_1(s, Y_s^1, Z_s^1) - f_1(s, Y_s^2, Z_s^1))l(\hat{Y}_s) + a_s^\varepsilon \hat{Y}_s,$$

where $a_s^\varepsilon = (1 - l(\hat{Y}_s))(f_1(s, Y_s^1, Z_s^1) - f_1(s, Y_s^2, Z_s^1))\hat{Y}_s^{-1} \in M_G^2(0, T)$ such that $|a_s^\varepsilon| \leq L$. It is easy to verify that

$$|(f_1(s, Y_s^1, Z_s^1) - f_1(s, Y_s^2, Z_s^1))l(\hat{Y}_s)| \leq L|\hat{Y}_s|l(\hat{Y}_s) \leq 2L\varepsilon.$$

Thus we can get

$$\hat{f}_s = a_s^\varepsilon \hat{Y}_s + b_s^\varepsilon \hat{Z}_s + m_s - m_s^\varepsilon, \quad \hat{g}_s = c_s^\varepsilon \hat{Y}_s + d_s^\varepsilon \hat{Z}_s + n_s - n_s^\varepsilon,$$

where $|m_s^\varepsilon| \leq 4L\varepsilon$, $|n_s^\varepsilon| \leq 4L\varepsilon$, $m_s = f_1(s, Y_s^2, Z_s^2) - f_2(s, Y_s^2, Z_s^2) \geq 0$ and $n_s = g_1(s, Y_s^2, Z_s^2) - g_2(s, Y_s^2, Z_s^2) \geq 0$. By Theorem 3.2, in the extended space,

we have

$$\begin{aligned}
& \hat{Y}_t + K_t^2 \\
&= (X_t^\varepsilon)^{-1} \hat{\mathbb{E}}_t^{\tilde{G}}[X_T^\varepsilon(\hat{\xi} + K_T^2) + \int_t^T (m_s - m_s^\varepsilon - a_s^\varepsilon K_s^2) X_s^\varepsilon ds \\
&+ \int_t^T (n_s - n_s^\varepsilon - c_s^\varepsilon K_s^2) X_s^\varepsilon d\langle B \rangle_s] \\
&\geq (X_t^\varepsilon)^{-1} \hat{\mathbb{E}}_t^{\tilde{G}}[X_T^\varepsilon K_T^2 - \int_t^T (m_s^\varepsilon + a_s^\varepsilon K_s^2) X_s^\varepsilon ds - \int_t^T (n_s^\varepsilon + c_s^\varepsilon K_s^2) X_s^\varepsilon d\langle B \rangle_s] \\
&\geq (X_t^\varepsilon)^{-1} \{ \hat{\mathbb{E}}_t^{\tilde{G}}[X_T^\varepsilon K_T^2 - \int_t^T a_s^\varepsilon K_s^2 X_s^\varepsilon ds - \int_t^T c_s^\varepsilon K_s^2 X_s^\varepsilon d\langle B \rangle_s] \\
&- \hat{\mathbb{E}}_t^{\tilde{G}}[\int_t^T m_s^\varepsilon X_s^\varepsilon ds + \int_t^T n_s^\varepsilon X_s^\varepsilon d\langle B \rangle_s] \},
\end{aligned}$$

where $\{X_t^\varepsilon\}_{t \in [0, T]}$ is the solution of the following \tilde{G} -SDE:

$$X_t^\varepsilon = 1 + \int_0^t a_s^\varepsilon X_s^\varepsilon ds + \int_0^t c_s^\varepsilon X_s^\varepsilon d\langle B \rangle_s + \int_0^t d_s^\varepsilon X_s^\varepsilon dB_s + \int_0^t b_s^\varepsilon X_s^\varepsilon d\tilde{B}_s.$$

By Theorem 3.2 and Proposition 3.5, we get

$$(X_t^\varepsilon)^{-1} \hat{\mathbb{E}}_t^{\tilde{G}}[X_T^\varepsilon K_T^2 - \int_t^T a_s^\varepsilon K_s^2 X_s^\varepsilon ds - \int_t^T c_s^\varepsilon K_s^2 X_s^\varepsilon d\langle B \rangle_s] = K_t^2.$$

Thus

$$\hat{Y}_t \geq -4L\varepsilon (X_t^\varepsilon)^{-1} \hat{E}^{\tilde{G}}[\int_t^T |X_s^\varepsilon| ds + \int_t^T |X_s^\varepsilon| d\langle B \rangle_s],$$

which complete the proof by letting $\varepsilon \rightarrow 0$. \square

Theorem 3.7 Let $(Y_t^i, Z_t^i, K_t^i)_{t \leq T}$, $i = 1, 2$, be the solutions of the following G -BSDEs:

$$Y_t^i = \xi^i + \int_t^T f_i(s) ds + \int_t^T g_i(s) d\langle B \rangle_s + V_T^i - V_t^i - \int_t^T Z_s^i dB_s - (K_T^i - K_t^i),$$

where $f_i(s) = f_i(s, Y_s^i, Z_s^i)$, $g_i(s) = g_i(s, Y_s^i, Z_s^i)$, $\xi^i \in L_G^\beta(\Omega_T)$, f_i, g_i satisfy (H1) and (H2), $(V_t^i)_{t \leq T}$ are RCLL processes such that $\hat{\mathbb{E}}[\sup_{t \in [0, T]} |V_t^i|^\beta] < \infty$ with $\beta > 1$. If $\xi^1 \geq \xi^2$, $f_1 \geq f_2$, $g_1 \geq g_2$, $V_t^1 - V_t^2$ is an increasing process, then $Y_t^1 \geq Y_t^2$.

Proof. The proof is similar to that of Theorem 3.6. \square

Remark 3.8 If f_i, g_i , $i = 1, 2$, do not contain Z , we get the following special G -BSDEs:

$$Y_t^i = \hat{\mathbb{E}}_t[\xi^i + \int_t^T f_i(s, Y_s^i) ds + \int_t^T g_i(s, Y_s^i) d\langle B \rangle_s].$$

The same as in Remark 3.4, here we suppose that $\xi \in L_G^1(\Omega)$, $\{f_i(s, y)\}_{s \in [0, T]} \in M_G^1(0, T)$ and $\{g_i(s, y)\}_{s \in [0, T]} \in M_G^1(0, T)$ for each $y \in \mathbb{R}$, f_i and g_i satisfy the Lipschitz condition with respect to y . The comparison theorem still holds for this case.

In the following, we give an example to show that the strict comparison theorem does not hold.

Example 3.9 We consider the simplest G -BSDE:

$$Y_t = \xi - \int_t^T Z_s dB_s - (K_T - K_t),$$

the solution $Y_t = \hat{\mathbb{E}}_t[\xi]$, $t \in [0, T]$. Let $\xi^1 = 0$ and $\xi^2 = \langle B \rangle_T - \bar{\sigma}^2 T$. It is easy to verify that $\xi^1 \geq \xi^2$ and $\hat{\mathbb{E}}[\xi^1 - \xi^2] > 0$ for the case $\underline{\sigma} < \bar{\sigma}$. But $\hat{\mathbb{E}}[\xi^1] = \hat{\mathbb{E}}[\xi^2] = 0$.

We now give an application of comparison theorem.

Theorem 3.10 (Gronwall inequality) Let $(Y_t)_{t \leq T} \in S_G^1(0, T)$ satisfy

$$Y_t \leq \hat{\mathbb{E}}_t[\xi + \int_t^T f(s, Y_s) ds + \int_t^T g(s, Y_s) d\langle B \rangle_s],$$

where $\xi \in L_G^1(\Omega)$, $\{f(s, y)\}_{s \in [0, T]} \in M_G^1(0, T)$ and $\{g(s, y)\}_{s \in [0, T]} \in M_G^1(0, T)$ for each $y \in \mathbb{R}$, f and g satisfy the Lipschitz condition with respect to y , $f(\cdot, y_1) \leq f(\cdot, y_2)$ and $g(\cdot, y_1) \leq g(\cdot, y_2)$ for each $y_1 \leq y_2$. Then $Y_t \leq \tilde{Y}_t$, where $(\tilde{Y}_t)_{t \leq T}$ is the solution of the following G -BSDE:

$$\tilde{Y}_t = \hat{\mathbb{E}}_t[\xi + \int_t^T f(s, \tilde{Y}_s) ds + \int_t^T g(s, \tilde{Y}_s) d\langle B \rangle_s].$$

In particular, if $f(s, y) = a_s y + m_s$, $g(s, y) = c_s y + n_s$, where $a_s \geq 0$, $c_s \geq 0$, then

$$Y_t \leq (X_t)^{-1} \hat{\mathbb{E}}_t[X_T \xi + \int_t^T m_s X_s ds + \int_t^T n_s X_s d\langle B \rangle_s], \quad (3.8)$$

where $X_t = \exp(\int_0^t a_s ds + \int_0^t c_s d\langle B \rangle_s)$.

Proof. We set

$$\delta_t = \hat{\mathbb{E}}_t[\xi + \int_t^T f(s, Y_s) ds + \int_t^T g(s, Y_s) d\langle B \rangle_s] - Y_t \geq 0,$$

then

$$\begin{aligned} Y_t + \delta_t &= \hat{\mathbb{E}}_t[\xi + \int_t^T f(s, Y_s) ds + \int_t^T g(s, Y_s) d\langle B \rangle_s] \\ &= \hat{\mathbb{E}}_t[\xi + \int_t^T f(s, Y_s + \delta_s - \delta_s) ds + \int_t^T g(s, Y_s + \delta_s - \delta_s) d\langle B \rangle_s]. \end{aligned}$$

Thus $(Y_t + \delta_t)_{t \leq T}$ is the solution of the following G -BSDE:

$$\bar{Y}_t = \hat{\mathbb{E}}_t[\xi + \int_t^T f(s, \bar{Y}_s - \delta_s) ds + \int_t^T g(s, \bar{Y}_s - \delta_s) d\langle B \rangle_s].$$

By comparison theorem of G -BSDEs, we get $\bar{Y}_t \leq \tilde{Y}_t$. Thus $Y_t \leq \tilde{Y}_t$. By formula (3.6), we get (3.8). \square

4 Nonlinear Feynman-Kac Formula

In this section, we give the nonlinear Feynman-Kac Formula which was studied in Peng [24] for special type of G -BSDEs. Let $G : \mathbb{S}_d \rightarrow \mathbb{R}$ be a given monotonic and sublinear function such that $G(A) - G(B) \geq \underline{\sigma}^2 \text{tr}[A - B]$ for any $A \geq B$ and $B_t = (B_t^i)_{i=1}^d$ be the corresponding G -Brownian motion. We consider the following type of G -FBSDEs:

$$dX_s^{t,\xi} = b(s, X_s^{t,\xi})ds + h_{ij}(s, X_s^{t,\xi})d\langle B^i, B^j \rangle_s + \sigma_j(s, X_s^{t,\xi})dB_s^j, \quad X_t^{t,\xi} = \xi, \quad (4.1)$$

$$\begin{aligned} Y_s^{t,\xi} &= \Phi(X_T^{t,\xi}) + \int_s^T f(r, X_r^{t,\xi}, Y_r^{t,\xi}, Z_r^{t,\xi})dr + \int_s^T g_{ij}(r, X_r^{t,\xi}, Y_r^{t,\xi}, Z_r^{t,\xi})d\langle B^i, B^j \rangle_r \\ &\quad - \int_s^T Z_r^{t,\xi}dB_r - (K_T^{t,\xi} - K_s^{t,\xi}), \end{aligned} \quad (4.2)$$

where $b, h_{ij}, \sigma_j : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$, $\Phi : \mathbb{R}^n \rightarrow \mathbb{R}$, $f, g_{ij} : [0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$ are deterministic functions and satisfy the following conditions:

- (A1) $h_{ij} = h_{ji}$ and $g_{ij} = g_{ji}$ for $1 \leq i, j \leq d$;
- (A2) $b, h_{ij}, \sigma_j, f, g_{ij}$ are continuous in t ;
- (A3) There exist a positive integer m and a constant $L > 0$ such that

$$\begin{aligned} |b(t, x) - b(t, x')| + \sum_{i,j=1}^d |h_{ij}(t, x) - h_{ij}(t, x')| + \sum_{j=1}^d |\sigma_j(t, x) - \sigma_j(t, x')| &\leq L|x - x'|, \\ |\Phi(x) - \Phi(x')| &\leq L(1 + |x|^m + |x'|^m)|x - x'|, \end{aligned}$$

$$\begin{aligned} |f(t, x, y, z) - f(t, x', y', z')| + \sum_{i,j=1}^d |g_{ij}(t, x, y, z) - g_{ij}(t, x', y', z')| \\ \leq L[(1 + |x|^m + |x'|^m)|x - x'| + |y - y'| + |z - z'|]. \end{aligned}$$

We have the following estimates of G -SDEs which can be found in Chapter V in Peng [24].

Proposition 4.1 *Let $\xi, \xi' \in L_G^p(\Omega_t; \mathbb{R}^n)$ with $p \geq 2$. Then we have, for each $\delta \in [0, T - t]$,*

$$\begin{aligned} \hat{\mathbb{E}}_t[|X_{t+\delta}^{t,\xi} - X_{t+\delta}^{t,\xi'}|^p] &\leq C|\xi - \xi'|^p, \\ \hat{\mathbb{E}}_t[|X_{t+\delta}^{t,\xi}|^p] &\leq C(1 + |\xi|^p), \\ \hat{\mathbb{E}}_t[\sup_{s \in [t, t+\delta]} |X_s^{t,\xi} - \xi|^p] &\leq C(1 + |\xi|^p)\delta^{p/2}, \end{aligned}$$

where the constant C depends on L, G, p, n and T .

Proof. For convenience of the reader, we sketch the proof. It is easy to verify that $(X_s^{t,\xi})_{s \in [t,T]}, (X_s^{t,\xi'})_{s \in [t,T]} \in M_G^p(0, T; \mathbb{R}^n)$. By Remark 2.10, we can get

$$\begin{aligned}\hat{\mathbb{E}}_t[|X_{t+\delta}^{t,\xi} - X_{t+\delta}^{t,\xi'}|^p] &\leq C_1(|\xi - \xi'|^p + \hat{\mathbb{E}}_t[\int_t^{t+\delta} |X_s^{t,\xi} - X_s^{t,\xi'}|^p ds]) \\ &\leq C_1(|\xi - \xi'|^p + \int_t^{t+\delta} \hat{\mathbb{E}}_t[|X_s^{t,\xi} - X_s^{t,\xi'}|^p] ds),\end{aligned}$$

where the constant C_1 depends on L, G, p, n and T . By the Gronwall inequality, we obtain

$$\hat{\mathbb{E}}_t[|X_{t+\delta}^{t,\xi} - X_{t+\delta}^{t,\xi'}|^p] \leq C_1 \exp(C_1 T) |\xi - \xi'|^p.$$

Then we get the first inequality. The other inequalities can be proved similarly. \square

Proposition 4.2 *For each $\xi, \xi' \in L_G^{4m+1}(\Omega_t; \mathbb{R}^n)$, we have*

$$\begin{aligned}|Y_t^{t,\xi} - Y_t^{t,\xi'}| &\leq C(1 + |\xi|^m + |\xi'|^m) |\xi - \xi'|, \\ |Y_t^{t,\xi}| &\leq C(1 + |\xi|^{m+1}),\end{aligned}$$

where the constant C depends on L, G, n and T .

Proof. It follows from Proposition 2.15 and Proposition 4.1 that

$$\begin{aligned}|Y_t^{t,\xi} - Y_t^{t,\xi'}|^2 &\leq C_1 \{ \hat{\mathbb{E}}_t[(1 + |X_T^{t,\xi}|^m + |X_T^{t,\xi'}|^m)^2 |X_T^{t,\xi} - X_T^{t,\xi'}|^2] \\ &\quad + \int_t^T \hat{\mathbb{E}}_s[(1 + |X_s^{t,\xi}|^m + |X_s^{t,\xi'}|^m)^2 |X_s^{t,\xi} - X_s^{t,\xi'}|^2] ds \} \\ &\leq C_2(1 + |\xi|^{2m} + |\xi'|^{2m}) \{ (\hat{\mathbb{E}}_t[|X_T^{t,\xi} - X_T^{t,\xi'}|^4])^{1/2} \\ &\quad + \int_t^T (\hat{\mathbb{E}}_s[|X_s^{t,\xi} - X_s^{t,\xi'}|^4])^{1/2} ds \} \\ &\leq C_3(1 + |\xi|^{2m} + |\xi'|^{2m}) |\xi - \xi'|^2,\end{aligned}$$

where C_1, C_2 and C_3 depend on L, G, n and T . Thus we get $|Y_t^{t,\xi} - Y_t^{t,\xi'}| \leq C(1 + |\xi|^m + |\xi'|^m) |\xi - \xi'|$. By Proposition 2.14, we can get $|Y_t^{t,\xi}| \leq C(1 + |\xi|^{m+1})$ by using the similar analysis. \square

We are more interested in the case when $\xi = x \in \mathbb{R}^n$. We define

$$u(t, x) := Y_t^{t,x}, \quad (t, x) \in [0, T] \times \mathbb{R}^n.$$

By Proposition 4.2, we immediately have the following estimates:

$$\begin{aligned}|u(t, x) - u(t, x')| &\leq C(1 + |x|^m + |x'|^m) |x - x'|, \\ |u(t, x)| &\leq C(1 + |x|^{m+1}),\end{aligned}$$

where the constant C depends on L, G, n and T .

Remark 4.3 *It is important to note that $u(t, x)$ is a deterministic function of (t, x) , because $b, h_{ij}, \sigma_j, \Phi, f, g_{ij}$ are deterministic functions and $\tilde{B}_s := B_{t+s} - B_t$ is a G -Brownian motion.*

The following theorem plays a key role in proving the Feynman-Kac formula.

Theorem 4.4 *For each $\xi \in L_G^{4m+1}(\Omega_t; \mathbb{R}^n)$, we have*

$$u(t, \xi) = Y_t^{t, \xi}.$$

Proof. By Proposition 4.2, we only need to prove Theorem 4.4 for bounded $\xi \in L_G^{4m+1}(\Omega_t; \mathbb{R}^n)$. Thus for each $\varepsilon > 0$, we can choose a simple function

$$\eta^\varepsilon = \sum_{i=1}^N x_i I_{A_i},$$

where $(A_i)_{i=1}^N$ is a $\mathcal{B}(\Omega_t)$ -partition and $x_i \in \mathbb{R}^n$, such that $|\eta^\varepsilon - \xi| \leq \varepsilon$. It follows from Proposition 4.2 that

$$\begin{aligned} |Y_t^{t, \xi} - u(t, \eta^\varepsilon)| &= |Y_t^{t, \xi} - \sum_{i=1}^n u(t, x_i) I_{A_i}| \\ &= |Y_t^{t, \xi} - \sum_{i=1}^N Y_t^{t, x_i} I_{A_i}| \\ &= \sum_{i=1}^N |Y_t^{t, \xi} - Y_t^{t, x_i}| I_{A_i} \\ &\leq \sum_{i=1}^N C(1 + |\xi|^m) |\xi - x_i| I_{A_i} \\ &= C(1 + |\xi|^m) |\xi - \sum_{i=1}^N x_i I_{A_i}| \\ &\leq C(1 + |\xi|^m) \varepsilon, \end{aligned}$$

where the constant C depends on L , G , n and T . Noting that

$$|u(t, \xi) - u(t, \eta^\varepsilon)| \leq C(1 + |\xi|^m) |\xi - \eta^\varepsilon| \leq C(1 + |\xi|^m) \varepsilon,$$

we get $|Y_t^{t, \xi} - u(t, \xi)| \leq 2C(1 + |\xi|^m) \varepsilon$. Since ε can be arbitrarily small, we obtain $Y_t^{t, \xi} = u(t, \xi)$. \square

We now give the Feynman-Kac formula.

Theorem 4.5 *Let $u(t, x) := Y_t^{t, x}$ for $(t, x) \in [0, T] \times \mathbb{R}^n$. Then $u(t, x)$ is the unique viscosity solution of the following PDE:*

$$\begin{cases} \partial_t u + F(D_x^2 u, D_x u, u, x, t) = 0, \\ u(T, x) = \Phi(x), \end{cases} \quad (4.3)$$

where

$$\begin{aligned} F(D_x^2 u, D_x u, u, x, t) &= G(H(D_x^2 u, D_x u, u, x, t)) + \langle b(t, x), D_x u \rangle \\ &\quad + f(t, x, u, \langle \sigma_1(t, x), D_x u \rangle, \dots, \langle \sigma_d(t, x), D_x u \rangle), \end{aligned}$$

$$\begin{aligned} H_{ij}(D_x^2 u, D_x u, u, x, t) &= \langle D_x^2 u \sigma_i(t, x), \sigma_j(t, x) \rangle + 2 \langle D_x u, h_{ij}(t, x) \rangle \\ &\quad + 2g_{ij}(t, x, u, \langle \sigma_1(t, x), D_x u \rangle, \dots, \langle \sigma_d(t, x), D_x u \rangle). \end{aligned}$$

Proof. The uniqueness of viscosity solution of equation (4.3) can be found in Appendix C in Peng [24], we only prove that u is a viscosity solution of equation (4.3). By $Y_{t+\delta}^{t,x} = Y_{t+\delta}^{t+\delta, X_{t+\delta}^{t,x}}$ and Theorem 4.4, we get $Y_{t+\delta}^{t,x} = u(t+\delta, X_{t+\delta}^{t,x})$ for $\delta \in [0, T-t]$ and

$$\begin{aligned} Y_t^{t,x} = & u(t+\delta, X_{t+\delta}^{t,x}) + \int_t^{t+\delta} f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) dr \\ & + \int_t^{t+\delta} g_{ij}(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) d\langle B^i, B^j \rangle_r - \int_t^{t+\delta} Z_r^{t,x} dB_r - (K_{t+\delta}^{t,x} - K_t^{t,x}). \end{aligned}$$

Taking G -expectation, we get

$$u(t, x) = \hat{\mathbb{E}}[u(t+\delta, X_{t+\delta}^{t,x})] + \int_t^{t+\delta} f_r dr + \int_t^{t+\delta} g_r^{ij} d\langle B^i, B^j \rangle_r,$$

where $f_r = f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})$, $g_r^{ij} = g_{ij}(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})$. In order to prove that u is a viscosity solution, we first show that u is a continuous function. By Proposition 4.2, we know that $|u(t, x) - u(t, x')| \leq C(1 + |x|^m + |x'|^m)|x - x'|$. By Proposition 4.1 and Proposition 2.14, we have $\hat{\mathbb{E}}_t[|X_{t+\delta}^{t,x} - x|^2] \leq C(1 + |x|^2)\delta$ and $\hat{\mathbb{E}}_t[|Y_r^{t,x}|^2 + \int_t^T |Z_r^{t,x}|^2 dr] \leq C(1 + |x|^{2m+2})$, where C depends on L, G, n and T . Thus we get

$$\begin{aligned} & |u(t, x) - u(t+\delta, x)| \\ & \leq C\{(1 + |x|^m)(\hat{\mathbb{E}}[|X_{t+\delta}^{t,x} - x|^2])^{1/2} + (\hat{\mathbb{E}}[\int_t^T (|f_r|^2 + |g_r^{ij}|^2) dr])^{1/2} \delta^{1/2}\} \\ & \leq C(1 + |x|^{m+1})\delta^{1/2}. \end{aligned}$$

It follows that u is a continuous function. For any fixed $(t, x) \in (0, T) \times \mathbb{R}^n$, let $\psi \in C^{2,3}([0, T] \times \mathbb{R}^n)$ be such that $\psi \geq u$, $\psi(t, x) = u(t, x)$ and $|\partial_{t x_i}^2 \psi(t, x)| + |\partial_{x_i x_j} \psi(t, x)| + |\partial_{x_i x_j x_k}^3 \psi(t, x)| \leq C(1 + |x|^{m_1})$ for some $m_1 > 0$. Let $(\tilde{Y}, \tilde{Z}, \tilde{K})$ be the solution of G -BSDE (4.2) on $[t, t+\delta]$ with terminal condition $\psi(t+\delta, X_{t+\delta}^{t,x})$. Set $\hat{Y}_s^1 = \tilde{Y}_s - \psi(s, X_s^{t,x})$, $\hat{Z}_s^1 = \tilde{Z}_s - (\langle \sigma_1(s, X_s^{t,x}), D_x \psi(s, X_s^{t,x}) \rangle, \dots, \langle \sigma_d(s, X_s^{t,x}), D_x \psi(s, X_s^{t,x}) \rangle)$, $\hat{K}_s^1 = \tilde{K}_s$, applying Itô's formula to $\hat{Y}_s - \psi(s, X_s^{t,x})$, we obtain that $(\hat{Y}^1, \hat{Z}^1, \hat{K}^1)$ is the solution of the following G -BSDE:

$$\begin{aligned} \hat{Y}_s^1 = & \int_s^{t+\delta} F_1(r, X_r^{t,x}, \hat{Y}_r^1, \hat{Z}_r^1) dr + \int_s^{t+\delta} F_2^{ij}(r, X_r^{t,x}, \hat{Y}_r^1, \hat{Z}_r^1) d\langle B^i, B^j \rangle_r \\ & - \int_s^{t+\delta} \hat{Z}_r^1 dB_r - (\hat{K}_{t+\delta}^1 - \hat{K}_s^1), \end{aligned}$$

where

$$\begin{aligned} F_1(r, x, y, z) = & f(r, x, y + \psi(r, x), z + (\langle \sigma_1, D_x \psi \rangle, \dots, \langle \sigma_d, D_x \psi \rangle)(r, x)) \\ & + \partial_t \psi(r, x) + \langle b(r, x), D_x \psi(r, x) \rangle, \end{aligned}$$

$$\begin{aligned} F_2^{ij}(r, x, y, z) = & g_{ij}(r, x, y + \psi(r, x), z + (\langle \sigma_1, D_x \psi \rangle, \dots, \langle \sigma_d, D_x \psi \rangle)(r, x)) \\ & + \langle D_x \psi(r, x), h_{ij}(r, x) \rangle + \frac{1}{2} \langle D_x^2 \psi(r, x) \sigma_i(r, x), \sigma_j(r, x) \rangle. \end{aligned}$$

Let $(\hat{Y}, \hat{Z}, \hat{K})$ be the solution of the following G -BSDE:

$$\begin{aligned}\hat{Y}_s &= \int_s^{t+\delta} F_1(r, x, \hat{Y}_r, \hat{Z}_r) dr + \int_s^{t+\delta} F_2^{ij}(r, x, \hat{Y}_r, \hat{Z}_r) d\langle B^i, B^j \rangle_r \\ &\quad - \int_s^{t+\delta} \hat{Z}_r dB_r - (\hat{K}_{t+\delta} - \hat{K}_s).\end{aligned}$$

It is easy to check that $\hat{Z}_s = 0$, \hat{Y}_s is the solution of the following ODE:

$$\begin{aligned}\hat{Y}_s &= \int_s^{t+\delta} [F_1(r, x, \hat{Y}_r, 0) + 2G(F_2(r, x, \hat{Y}_r, 0))] dr, \\ \hat{K}_s &= \int_t^s F_2^{ij}(r, x, \hat{Y}_r, 0) d\langle B^i, B^j \rangle_r - \int_t^s 2G(F_2(r, x, \hat{Y}_r, 0)) dr,\end{aligned}$$

where $F_2(r, x, \hat{Y}_r, 0) = (F_2^{ij}(r, x, \hat{Y}_r, 0))_{i,j=1}^d$. By Proposition 2.15, we have for any fixed $p > 2$

$$\begin{aligned}|\hat{Y}_t^1 - \hat{Y}_t|^2 &\leq \hat{\mathbb{E}}[\sup_{s \in [t, t+\delta]} |\hat{Y}_s^1 - \hat{Y}_s|^2] \\ &\leq C\{\hat{\mathbb{E}}[\sup_{s \in [t, t+\delta]} \hat{\mathbb{E}}_s[(\int_t^{t+\delta} \hat{F}_r dr)^p]]^{2/p} + \hat{\mathbb{E}}[\sup_{s \in [t, t+\delta]} \hat{\mathbb{E}}_s[(\int_t^{t+\delta} \hat{F}_r dr)^p]]\},\end{aligned}$$

where $\hat{F}_r = |F_1(r, X_r^{t,x}, \hat{Y}_r, 0) - F_1(r, x, \hat{Y}_r, 0)| + \sum_{i,j=1}^d |F_2^{ij}(r, X_r^{t,x}, \hat{Y}_r, 0) - F_2^{ij}(r, x, \hat{Y}_r, 0)|$. It is easy to verify that there exists a constant $m_2 > 0$ such that

$$\hat{F}_r \leq C(1 + |x|^{m_2} + |X_r^{t,x}|^{m_2})|X_r^{t,x} - x|.$$

Then by Theorem 2.13 in [7] and Proposition 4.1 we can deduce that $|\hat{Y}_t^1 - \hat{Y}_t| \leq C(1 + |x|^{m_2+2})\delta^{\frac{3}{2}}$. By comparison theorem of G -BSDEs, we know that $\hat{Y}_t \geq u(t, x)$, that is $\hat{Y}_t^1 \geq 0$. Then we get

$$-C(1 + |x|^{m_2+2})\delta^{1/2} \leq \delta^{-1}\hat{Y}_t = \delta^{-1} \int_t^{t+\delta} [F_1(r, x, \hat{Y}_r, 0) + 2G(F_2(r, x, \hat{Y}_r, 0))] dr.$$

Letting $\delta \rightarrow 0$, we obtain $F_1(t, x, 0, 0) + 2G(F_2(t, x, 0, 0)) \geq 0$, which implies that u is a viscosity subsolution. Similarly we can prove that u is a viscosity supersolution. \square

5 Girsanov transformation

5.1 Nonlinear expectations generated by G -BSDEs

For simplicity, we consider the following G -BSDE driven by 1-dimensional G -Brownian motion. The results still hold for the case $d > 1$.

$$\begin{aligned}Y_t^{T,\xi} &= \xi + \int_t^T f(s, Y_s^{T,\xi}, Z_s^{T,\xi}) ds + \int_t^T g(s, Y_s^{T,\xi}, Z_s^{T,\xi}) d\langle B \rangle_s \\ &\quad - \int_t^T Z_s^{T,\xi} dB_s - (K_T^{T,\xi} - K_t^{T,\xi}),\end{aligned}\tag{5.1}$$

where f and g satisfy the Lipschitz condition. We further suppose that $f(s, y, 0) = g(s, y, 0) = 0$. We define, for each $\xi \in L_G^\beta(\Omega_T)$ with $\beta > 1$,

$$\tilde{\mathbb{E}}_{t,T}[\xi] := Y_t^{T,\xi}.$$

It is easy to verify that for each $T_1 < T_2$ and $\xi \in L_G^\beta(\Omega_{T_1})$ with $\beta > 1$, $\tilde{\mathbb{E}}_{t,T_1}[\xi] = \tilde{\mathbb{E}}_{t,T_2}[\xi]$. Thus we use the notation $\tilde{\mathbb{E}}_t[\xi]$.

Theorem 5.1 *We have*

- (1) For each $\xi^1 \geq \xi^2$, we have $\tilde{\mathbb{E}}_t[\xi^1] \geq \tilde{\mathbb{E}}_t[\xi^2]$;
- (2) For each $\xi \in L_G^\beta(\Omega_t)$ with $\beta > 1$, $\tilde{\mathbb{E}}_t[\xi] = \xi$;
- (3) $\tilde{\mathbb{E}}_t[\tilde{\mathbb{E}}_s[\xi]] = \tilde{\mathbb{E}}_{t \wedge s}[\xi]$;
- (4) If f and g are positively homogeneous, then for each $\lambda_t \in L_G^\infty(\Omega_t)$, we have $\tilde{\mathbb{E}}_t[\lambda_t \xi] = \lambda_t \tilde{\mathbb{E}}_t[\xi]$;
- (5) If f and g are subadditive, then $\tilde{\mathbb{E}}_t[\xi^1 + \xi^2] \leq \tilde{\mathbb{E}}_t[\xi^1] + \tilde{\mathbb{E}}_t[\xi^2]$;
- (6) If f and g are convex, then $\tilde{\mathbb{E}}_t[\lambda_t \xi^1 + (1 - \lambda_t) \xi^2] \leq \lambda_t \tilde{\mathbb{E}}_t[\xi^1] + (1 - \lambda_t) \tilde{\mathbb{E}}_t[\xi^2]$ for each $\lambda_t \in L_G^\infty(\Omega_t)$ and $\lambda_t \in [0, 1]$;
- (7) For each $\xi \in L_G^1(\Omega_t; \mathbb{R}^m)$, $\eta \in L_G^1(\Omega_T; \mathbb{R}^n)$, $\Phi \in C_{b.Lip}(\mathbb{R}^{m+n})$, we have

$$\tilde{\mathbb{E}}_t[\Phi(\xi, \eta)] = \tilde{\mathbb{E}}_t[\Phi(x, \eta)]_{x=\xi}.$$

- (8) Let K be a decreasing G -martingale with $K_T \in L_G^\alpha(\Omega_T)$ for some $\alpha > 1$. Then we have

$$\tilde{\mathbb{E}}_s[K_t] = K_s, \text{ for any } s \leq t.$$

Proof. It is easy to get (1)-(3). (8) is straightforward from Proposition 3.5. First we prove (6). (4) and (5) can be proved similarly. Let (Y^i, Z^i, K^i) , $i = 1, 2$, be the solutions of G -BSDE (5.1) corresponding to ξ^i . We have for $r \in [t, T]$

$$\tilde{Y}_r = \tilde{\xi} + \int_r^T \tilde{f}_s ds + \int_r^T \tilde{g}_s d\langle B \rangle_s - \tilde{K}_T^2 + \tilde{K}_r^2 - \int_r^T \tilde{Z}_s dB_s - (\tilde{K}_T^1 - \tilde{K}_r^1),$$

where $\tilde{Y}_r = \lambda_t Y_r^1 + (1 - \lambda_t) Y_r^2$, $\tilde{\xi} = \lambda_t \xi^1 + (1 - \lambda_t) \xi^2$, $\tilde{f}_s = \lambda_t f(s, Y_s^1, Z_s^1) + (1 - \lambda_t) f(s, Y_s^2, Z_s^2)$, $\tilde{g}_s = \lambda_t g(s, Y_s^1, Z_s^1) + (1 - \lambda_t) g(s, Y_s^2, Z_s^2)$, $\tilde{Z}_s = \lambda_t Z_s^1 + (1 - \lambda_t) Z_s^2$, $\tilde{K}_r^1 = \lambda_t K_r^1$, $\tilde{K}_r^2 = (1 - \lambda_t) K_r^2$. By the convexity of f and g , we get $\tilde{f}_s \geq f(s, \tilde{Y}_s, \tilde{Z}_s)$ and $\tilde{g}_s \geq g(s, \tilde{Y}_s, \tilde{Z}_s)$. Note that $-\tilde{K}_r$ is an increasing process, then by Theorem 3.7 we obtain $\tilde{\mathbb{E}}_t[\xi] \leq \tilde{Y}_t$, which implies (6).

We now prove (7). For each given $n \in \mathbb{N}$, we can choose $A_i^n \in \mathcal{B}(\mathbb{R}^m)$, $i = 1, \dots, k_n$, such that $A_i^n \cap A_j^n = \emptyset$ for $i \neq j$, $\cup_{i=1}^{k_n} A_i^n = \mathbb{R}^m$, $\{x : |x| \leq n\} \subset \cup_{i=1}^{k_n-1} A_i^n$ and $\lambda(A_i^n) \leq 1/n$ for $i \leq k_n - 1$, where $\lambda(A_i^n)$ denote the diameter of

A_i . Let $x_i^n \in A_i^n$, by Proposition 2.15, we have

$$\begin{aligned}
& \left| \sum_{i=1}^{k_n} \tilde{\mathbb{E}}_t[\Phi(x_i^n, \eta)] I_{A_i^n}(\xi) - \tilde{\mathbb{E}}_t[\Phi(\xi, \eta)] \right|^2 \\
&= \sum_{i=1}^{k_n} I_{A_i^n}(\xi) |\tilde{\mathbb{E}}_t[\Phi(x_i^n, \eta)] - \tilde{\mathbb{E}}_t[\Phi(\xi, \eta)]|^2 \\
&\leq C \sum_{i=1}^{k_n} I_{A_i^n}(\xi) \hat{\mathbb{E}}_t[|\Phi(x_i^n, \eta) - \Phi(\xi, \eta)|^2] \\
&= C \hat{\mathbb{E}}_t \left[\sum_{i=1}^{k_n} I_{A_i^n}(\xi) |\Phi(x_i^n, \eta) - \Phi(\xi, \eta)|^2 \right],
\end{aligned}$$

where C is a constant independent of n . Note that

$$\sum_{i=1}^{k_n} I_{A_i^n}(\xi) |\Phi(x_i^n, \eta) - \Phi(\xi, \eta)|^2 \leq \frac{L^2}{n^2} + 4\|\Phi\|_\infty^2 I_{[|\xi| > n]},$$

where L is the Lipschitz constant of Φ , then we get

$$\begin{aligned}
& \hat{\mathbb{E}} \left[\sum_{i=1}^{k_n} \tilde{\mathbb{E}}_t[\Phi(x_i^n, \eta)] I_{A_i^n}(\xi) - \tilde{\mathbb{E}}_t[\Phi(\xi, \eta)] \right]^2 \\
&\leq C \hat{\mathbb{E}} \left[\frac{L^2}{n^2} + 4\|\Phi\|_\infty^2 I_{[|\xi| > n]} \right] \\
&\leq C \left\{ \frac{L^2}{n^2} + \frac{4\|\Phi\|_\infty^2}{n} \hat{\mathbb{E}}[|\xi|] \right\} \rightarrow 0.
\end{aligned}$$

On the other hand, by Proposition 2.15, we know that there exists a constant $C > 0$ such that

$$|\tilde{\mathbb{E}}_t[\Phi(x, \eta)] - \tilde{\mathbb{E}}_t[\Phi(y, \eta)]| \leq C|x - y| \quad \text{for } x, y \in \mathbb{R}^m.$$

Thus

$$\begin{aligned}
& \hat{\mathbb{E}} \left[\sum_{i=1}^{k_n} \tilde{\mathbb{E}}_t[\Phi(x_i^n, \eta)] I_{A_i^n}(\xi) - \tilde{\mathbb{E}}_t[\Phi(x, \eta)]_{x=\xi} \right]^2 \\
&= \hat{\mathbb{E}} \left[\sum_{i=1}^{k_n} I_{A_i^n}(\xi) |\tilde{\mathbb{E}}_t[\Phi(x_i^n, \eta)] - \tilde{\mathbb{E}}_t[\Phi(x, \eta)]_{x=\xi}|^2 \right] \\
&\leq \hat{\mathbb{E}} \left[\frac{C^2}{n^2} + 4\|\Phi\|_\infty^2 I_{[|\xi| > n]} \right] \\
&\leq \frac{C^2}{n^2} + \frac{4\|\Phi\|_\infty^2}{n} \hat{\mathbb{E}}[|\xi|] \rightarrow 0,
\end{aligned}$$

which implies $\tilde{\mathbb{E}}_t[\Phi(\xi, \eta)] = \tilde{\mathbb{E}}_t[\Phi(x, \eta)]_{x=\xi}$. \square

5.2 Girsanov transformation

We first consider the following G -BSDE driven by 1-dimensional G -Brownian motion:

$$Y_t = \xi + \int_t^T b_s Z_s ds + \int_t^T d_s Z_s d\langle B \rangle_s - \int_t^T Z_s dB_s - (K_T - K_t),$$

where $(b_t)_{t \leq T}$ and $(d_t)_{t \leq T}$ are bounded processes. For each $\xi \in L_G^\beta(\Omega_T)$ with $\beta > 1$, define

$$\tilde{\mathbb{E}}_t[\xi] = Y_t.$$

By Theorem 5.1, we know that $\tilde{\mathbb{E}}_t[\cdot]$ is a consistent sublinear expectation.

Theorem 5.2 (*Girsanov Theorem*) *Let $(b_t)_{t \leq T}$ and $(d_t)_{t \leq T}$ be bounded processes. Then $\bar{B}_t := B_t - \int_0^t b_s ds - \int_0^t d_s d\langle B \rangle_s$ is a G -Brownian motion under $\tilde{\mathbb{E}}$.*

Proof. We only need to show that for each $\Phi \in C_{b.Lip}(\mathbb{R}^n)$, $t_1 < \dots < t_n$,

$$\tilde{\mathbb{E}}[\Phi(\bar{B}_{t_1}, \bar{B}_{t_2} - \bar{B}_{t_1}, \dots, \bar{B}_{t_n} - \bar{B}_{t_{n-1}})] = \hat{\mathbb{E}}[\Phi(B_{t_1}, B_{t_2} - B_{t_1}, \dots, B_{t_n} - B_{t_{n-1}})].$$

Step 1. We consider the case $b_s \equiv b$ and $d_s \equiv d$. For each $\varphi \in C_{b.Lip}(\mathbb{R})$, we define

$$\tilde{u}(t, x) = \tilde{\mathbb{E}}[\varphi(x + \bar{B}_t)].$$

Set $u(t, x) = \tilde{u}(T - t, x)$ for fixed $T > 0$, by Theorem 4.5, we obtain u satisfies the following PDE:

$$\partial_t u - b \partial_x u + b \partial_x u + 2G(-d \partial_x u + \frac{1}{2} \partial_{xx}^2 u + d \partial_x u) = 0, u(T, x) = \varphi(x),$$

i.e. $\partial_t u + G(\partial_{xx}^2 u) = 0$, $u(T, x) = \varphi(x)$. Thus $\tilde{\mathbb{E}}[\varphi(\bar{B}_t)] = \hat{\mathbb{E}}[\varphi(B_t)]$ for any $t \geq 0$, $\varphi \in C_{b.Lip}(\mathbb{R})$.

Step 2. We consider the case $b_s^n = \sum_{i=0}^{n-1} \xi_i I_{[t_i^n, t_{i+1}^n)}(s)$, $d_s^n = \sum_{i=0}^{n-1} \eta_i I_{[t_i^n, t_{i+1}^n)}(s)$, where $\xi_i, \eta_i \in Lip(\Omega_{t_i^n})$. For each $\varphi \in C_{b.Lip}(\mathbb{R})$, we have

$$\tilde{\mathbb{E}}[\varphi(\bar{B}_{t_{i+1}^n})] = \tilde{\mathbb{E}}[\varphi(\bar{B}_{t_i^n} + B_{t_{i+1}^n} - B_{t_i^n} - \xi_i(t_{i+1}^n - t_i^n) - \eta_i(\langle B \rangle_{t_{i+1}^n} - \langle B \rangle_{t_i^n}))].$$

By (7) in Theorem 5.1, we get

$$\begin{aligned} & \tilde{\mathbb{E}}[\varphi(\bar{B}_{t_{i+1}^n})] \\ &= \tilde{\mathbb{E}}[\varphi(x + B_{t_{i+1}^n} - B_{t_i^n} - b(t_{i+1}^n - t_i^n) - d(\langle B \rangle_{t_{i+1}^n} - \langle B \rangle_{t_i^n}))]_{x=\bar{B}_{t_i^n}, b=\xi_i, d=\eta_i} \\ &= \tilde{\mathbb{E}}[\hat{\mathbb{E}}[\varphi(x + B_{t_{i+1}^n} - B_{t_i^n})]_{x=\bar{B}_{t_i^n}}]. \end{aligned}$$

Repeat this process, we obtain $\tilde{\mathbb{E}}[\varphi(\bar{B}_{t_{i+1}^n})] = \hat{\mathbb{E}}[\varphi(B_{t_{i+1}^n})]$. Similarly, we can get

$$\tilde{\mathbb{E}}[\Phi(\bar{B}_{t_1}, \bar{B}_{t_2} - \bar{B}_{t_1}, \dots, \bar{B}_{t_n} - \bar{B}_{t_{n-1}})] = \hat{\mathbb{E}}[\Phi(B_{t_1}, B_{t_2} - B_{t_1}, \dots, B_{t_n} - B_{t_{n-1}})].$$

Step 3. For general bounded processes (b_t) and (d_t) , we can choose uniformly bounded processes $(b_t^n), (d_t^n) \in M_G^{2,0}(0, T)$ such that $\|b^n - b\|_{M_G^2} + \|d^n - d\|_{M_G^2} \rightarrow 0$. By Proposition 2.15, we obtain the result by letting $n \rightarrow \infty$. \square

Remark 5.3 If $b_s = 0$, we know by Remark 3.3

$$\tilde{\mathbb{E}}_t[\xi] = \hat{\mathbb{E}}_t[\xi \exp(\int_t^T d_s dB_s - \frac{1}{2} \int_t^T |d_s|^2 d\langle B \rangle_s)].$$

This type of Girsanov transformation was studied in [31, 11], but here we give a simple proof. If $b_s \neq 0$, we know by Theorem 3.2

$$\begin{aligned} \tilde{\mathbb{E}}_t[\xi] = & \hat{\mathbb{E}}_t^{\tilde{G}}[\xi \exp(\int_t^T d_s dB_s - \frac{1}{2} \int_t^T |d_s|^2 d\langle B \rangle_s - \int_t^T b_s d_s ds \\ & + \int_t^T b_s d\tilde{B}_s - \frac{1}{2} \int_t^T |b_s|^2 d\langle \tilde{B} \rangle_s)], \end{aligned}$$

where (B, \tilde{B}) is an auxiliary extended \tilde{G} -Brownian motion and

$$\tilde{G}(A) = \frac{1}{2} \sup_{\underline{\sigma}^2 \leq v \leq \bar{\sigma}^2} \text{tr} \left[A \begin{bmatrix} v & 1 \\ 1 & v^{-1} \end{bmatrix} \right], \quad A \in \mathbb{S}_2.$$

We now consider the Girsanov transformation for the case $d > 1$. Let $B_t = (B_t^i)_{i=1}^d$ be a d -dimensional G -Brownian motion. We consider the following G -BSDE:

$$Y_t = \xi + \int_t^T b_s Z_s ds + \int_t^T d_s^{ij} Z_s d\langle B^i, B^j \rangle_s - \int_t^T Z_s dB_s - (K_T - K_t),$$

where $(b_t)_{t \leq T}$ and $(d_t^{ij})_{t \leq T}$ are \mathbb{R}^d -valued bounded processes. By Theorem 5.1, $\tilde{\mathbb{E}}_t[\xi] := Y_t$ is a consistent sublinear expectation.

Theorem 5.4 (*Girsanov Theorem*) Let $(b_t)_{t \leq T}$ and $(d_t^{ij})_{t \leq T}$ be \mathbb{R}^d -valued bounded processes. Then $\tilde{B}_t := B_t - \int_0^t b_s ds - \int_0^t d_s^{ij} d\langle B^i, B^j \rangle_s$ is a d -dimensional G -Brownian motion under $\tilde{\mathbb{E}}$.

Proof. The proof is similar to Theorem 5.2. \square

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